



# LLR Approximation for Wireless Channels Based on Taylor Series and Its Application to BICM with LDPC Codes

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## Abstract

A new approach for the approximation of the channel log-likelihood ratio (LLR) for wireless channels based on Taylor series is proposed. The approximation is applied to the uncorrelated flat Rayleigh fading channel with unknown channel state information at the receiver. It is shown that the proposed approximation greatly simplifies the calculation of channel LLRs, and yet provides results almost identical to those based on the exact calculation of channel LLRs. The results are obtained in the context of bit-interleaved coded modulation (BICM) schemes with low-density parity-check (LDPC) codes, and include threshold calculations and error rate performance of finite-length codes. Compared to the existing approximations, the proposed method is either significantly less complex, or considerably more accurate.

## Index Terms

Log-likelihood ratio (LLR), LLR approximation, wireless channels, flat fading, Rayleigh fading, uncorrelated Rayleigh fading, bit-interleaved coded modulation (BICM), low-density parity-check (LDPC) codes, iterative decoding, decoding threshold, SNR threshold.

## I. INTRODUCTION

In binary transmission over a wireless channel, the derivation of channel log-likelihood ratios (LLR) is often needed at the receiver for the detection and/or decoding of information. The channel LLR is sometimes referred to as *soft information*, and its availability can improve the performance of the detection/decoding schemes significantly. The channel LLR values depend on the channel output, the noise power and the fading characteristics as well as the amount of channel state information (CSI) available at the receiver. In practical systems, however, acquiring the CSI would require extra bandwidth for the transmission of pilot symbols and extra complexity at the receiver for the channel estimation. In certain scenarios, this may not be desirable. It is thus important to derive the LLR at the absence of CSI. In particular, low-complexity approximations of LLR are of great practical importance. One should note that even if efforts are made to estimate the channel and to attain the CSI, there always exist errors in the estimation process, which in turn results in imperfect CSI at the receiver. There has been thus literature on the study of the effect of imperfect CSI on the performance of transmission schemes (see, e.g., [11]), and on the design of schemes which are robust to such imperfections (see, e.g., [12]).

At the absence of CSI at the receiver, the relationship between the channel LLR,  $L$ , on one hand, and the channel output  $Y$ , and the noise power  $\sigma_n^2$ , on the other hand, is complex. This can significantly increase the complexity of a detection/decoding process which relies on the calculation of channel LLR values. In addition, the complex relationship  $L(Y, \sigma_n^2)$ , can impede the analysis and the design of transmission schemes over wireless channels which depend on the calculation of the probability density function (pdf) of  $L$  as the starting point. One example is the application of *density evolution* [9] to the analysis and the design of coded schemes. Motivated by these, much research has been devoted to the approximations of channel LLR values for wireless channels; see, e.g., [6], [13], [14], [15] and the references therein. In particular, linear or piece-wise linear approximations of  $L$  as a function of  $Y$  have received special attention due to their utmost simplicity.

To analyze and design binary low-density parity-check (LDPC) codes for binary phase shift keying (BPSK) transmission over flat Rayleigh fading channels in the absence of CSI, Hou *et al.* [6] proposed

the following linear approximation of the channel LLR:

$$\hat{L}_A = \frac{2}{\sigma_n^2} E(A)Y \triangleq \alpha_A Y, \quad (1)$$

where  $E(A)$  is the expected value of the channel gain. Though very simple, this approximation is not very accurate and performs rather poorly compared to true LLR values [14]. A more accurate linear approximation of LLR was recently proposed in [14] as

$$\hat{L}_{\hat{C}} = \alpha_{\hat{C}} Y, \quad (2)$$

where

$$\alpha_{\hat{C}} = \arg \max_{\alpha} \left\{ 1 - \int_{-\infty}^{\infty} \log_2(1 + e^{-\hat{l}}) f_{\hat{L}}(\hat{l}) d\hat{l} \right\}. \quad (3)$$

In (3),  $f_{\hat{L}}(\hat{l})$  is the pdf of the approximate linear LLR parameterized by  $\alpha$  according to  $\hat{L} = \alpha Y$ . It is shown in [14] that the approximation in (2) provides considerable improvement compared to (1) for the BPSK modulation and results in performances very close to those of the true LLR calculation. The calculation of  $\alpha_{\hat{C}}$  is however, much more complex than that of  $\alpha_A$  and requires solving the convex optimization problem of (3) using numerical techniques.

The general approach of [14] based on the formulation of (2) and (3) was then generalized to non-binary modulation schemes in [15], where piece-wise linear approximations of LLR for 8-PAM and 16-QAM constellations were derived. These results were then used to evaluate the performance of LDPC-coded BICM schemes. It was demonstrated [15] that for the tested LDPC codes, the gap in the performance of 8-PAM for true LLR calculations and the approximations is rather small (a few hundredths of a dB). This gap however, is larger for 16-QAM (about 0.2 dB).

In this paper, we propose to use the Taylor series of the channel LLR as the method of approximation. Although many of our results are in principle applicable to a variety of fading channel models, in this work, we only consider the uncorrelated flat Rayleigh fading channel. For the BPSK transmission over this channel with no CSI, we derive the Taylor series analytically and demonstrate that by using only the first term of the series, one can obtain an analytical linear approximation which is almost as simple as (1) and yet is practically as accurate as (2). By using the first two terms of the series, we derive a more accurate analytical non-linear approximation of the channel LLR and obtain performance improvements compared to the approximation of (2).

For non-binary modulations, we derive piece-wise linear and non-linear approximations of channel LLRs based on the Taylor series. Compared to the piece-wise linear approximation of [15], our approach is simpler, both conceptually and complexity-wise. Moreover, our approach can be easily extended from

piece-wise linear to non-linear approximations. This however, may not be simply achievable for the approach of [15], where the linearity has an important consequence of making the optimization problem convex and thus tractable. Performance-wise, we demonstrate that for the 8-PAM constellation in an LDPC-coded BICM scheme, our piece-wise linear approximation performs as good as the approximation of [15] and very close to true LLR calculations. We expect this to be the case also for other one-dimensional constellations. For two-dimensional constellations such as 16-QAM however, our second order Taylor series approximation of LLR outperforms the approximation of [15] handily, and still performs very close to true LLR calculations.

It is important to note that the complexity of calculating the approximate LLR is particularly important for two-dimensional constellations. While one might argue that such computations can be performed off-line and the results can be stored at look-up tables for different values of noise power and received signal values, such tables will have to be three dimensional for two-dimensional constellations, thus requiring much larger storage. If storage is constrained, one may have to perform the calculations on-line. Another point worth emphasizing is the importance of the computational complexity of finding the pdf of the LLR (or its approximations) in the process of analyzing or designing a transmission scheme using techniques such as density evolution [9]. These techniques, which are used for iterative coding/processing schemes (also known as *message-passing* schemes), are based on tracking the pdf of messages throughout iterations, starting from the pdf of the channel LLR. This is usually performed multiple times for different values of the channel signal-to-noise ratio (SNR) to find the infimum value of SNR for which the algorithm succeeds (probability of error tends to zero as the number of iterations tends to infinity). This infimum value of the channel SNR is called the *threshold*. One example of such analysis, is to find the threshold of an ensemble of LDPC codes [9]. One thus needs to find the pdf of the channel LLR many times for different SNR values in this process. The number of times such computations have to be repeated increases even further (by a significant margin) in a design process. Such a process is usually based on iteratively optimizing different variables to achieve the best performance. For example, in the design of irregular LDPC code ensembles, degree distributions of variable nodes and check nodes are optimized to achieve the best threshold [10]. This design process usually includes an analysis loop which is repeated numerous times as the design variables are modified to converge to a local optimum.

The remainder of the paper is organized as follows: Section II is devoted to the fading channel model, LDPC codes, BICM scheme and the derivation of channel LLRs. In Section III, we present the Taylor series approximations of the channel LLR for uncorrelated flat Rayleigh fading channels. Simulation

results are presented in Section IV, and finally Section V concludes the paper.

## II. CHANNEL MODEL, BICM SCHEME, AND LDPC CODES

### A. Channel Model, BICM and LLR Approximations

Consider the following model of a flat fading channel:

$$Y_t = A_t X_t + Z_t, \quad (4)$$

where  $X_t$  and  $Y_t$  represent the channel input and output at time  $t$ , respectively;  $Z_t$  is the zero-mean (possibly complex) Gaussian noise with variance  $\sigma^2$  ( $2\sigma^2$  for two-dimensional constellations), and  $A_t \geq 0$  is the channel gain, both at time  $t$ . In this work, we assume that  $A_t$  has a normalized Rayleigh distribution, i.e.,  $p_{A_t}(a) = 2a e^{-a^2}$ , where  $p_{A_t}(a)$  is the pdf of  $A_t$ . We further assume that sequences  $\{X_t\}$ ,  $\{A_t\}$  and  $\{Z_t\}$  consist of independent and identically distributed (i.i.d) random variables. Moreover the three sequences are assumed to be independent of each other. This model is referred to as the *uncorrelated flat Rayleigh fading channel*.

At the transmitter, the information bit sequence is first mapped to the coded bit sequence by being passed through the LDPC encoder. The coded bit sequence is then partitioned into blocks of length  $m$ . Each block  $\mathbf{b}_k = \{b_k^1, b_k^2, \dots, b_k^m\}$  is then mapped to a signal  $X_k$ ,  $k = 1, \dots, M$ , from a Gray-labeled  $M$ -ary signal constellation  $\chi$  with  $M = 2^m$  signals. The block  $\mathbf{b}_k$  is referred to as the  $k$ th *symbol* corresponding to the  $k$ th signal, and  $b_k^i$  is the  $i$ th bit of the  $k$ th symbol (signal). Between the encoder and the modulator, an interleaver conventionally exists. This however may not be required for the LDPC codes, as the interleaver is inherent in the structure of these codes when the parity-check matrix is constructed randomly [8]. For  $M > 2$ , this setting is also known as bit-interleaved coded modulation (BICM). In this work, we assume *ideal* interleaving of the bits, which implies that the transmission of each symbol over the channel is equivalent to the transmission of its constituent bits over  $m$  parallel and independent memoryless binary-input channels. These channels are referred to as *bit-channels*. At the receiver, the LLR for each bit-channel is independently calculated.

In the channel model described above, assuming that the noise variance is known at the receiver, we will have two scenarios depending on the availability of  $A_t$  at the receiver:<sup>1</sup>

<sup>1</sup>In the rest of the paper, the time index  $t$  may be dropped since the distribution of random variables does not depend on  $t$ . Also, upper case and lower case variables are used to denote random variables and their values, respectively, e.g., random variable  $Y$  can take the value  $y$ .

- 1) *Known CSI*: In this case, the channel gain  $A_t$  is known at the receiver for every  $t$ . Thus the channel LLR of the  $i$ th bit,  $l^{(i)}$ , corresponding to the output  $y$  and the channel gain  $a$ , is given by

$$l^{(i)} = \log \frac{p(y|b^i(x) = 0, a)}{p(y|b^i(x) = 1, a)} = \log \frac{\sum_{x \in \chi_0^i} p(y|x, a)}{\sum_{x \in \chi_1^i} p(y|x, a)} \triangleq g_a^i(y), \quad (5)$$

where  $b^i(x)$ ,  $i \in \{1, \dots, m\}$ , is the  $i$ th bit of the signal  $x$ ,  $\chi_w^i$  is the subset of the signals  $x$  in  $\chi$  where  $b^i(x) = w$ ,  $w \in \{0, 1\}$ , and the conditional pdfs are given by  $p(y|x, a) = \frac{1}{2\pi\sigma^2} \exp(-\frac{\|y-ax\|^2}{2\sigma^2})$ , for two-dimensional signal constellations, or by  $\frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(y-ax)^2}{2\sigma^2})$  for one-dimensional constellations. For the case of BPSK modulation, the above formulation simplifies to:

$$l = \frac{2a}{\sigma^2} y. \quad (6)$$

- 2) *Unknown CSI*: In this case, the channel gain  $A_t$  is unavailable at the receiver. The channel LLR for the  $i$ th bit can then be calculated as

$$l^{(i)} = \log \frac{p(y|b^i(x) = 0)}{p(y|b^i(x) = 1)} = \log \frac{\sum_{x \in \chi_0^i} p(y|x)}{\sum_{x \in \chi_1^i} p(y|x)} \triangleq g^i(y), \quad (7)$$

where  $p(y|x) = \int_0^\infty \frac{1}{2\pi\sigma^2} \exp(-\frac{\|y-ax\|^2}{2\sigma^2}) p_A(a) da$ , for two-dimensional constellations, or  $p(y|x) = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(y-ax)^2}{2\sigma^2}) p_A(a) da$ , for one-dimensional constellations (including BPSK).

- **BPSK**

For BPSK modulation over normalized Rayleigh fading channels, Equation (7) reduces to [14]

$$l = \log \frac{\Phi(y/\sqrt{2\sigma^2(1+2\sigma^2)})}{\Phi(-y/\sqrt{2\sigma^2(1+2\sigma^2)})}, \quad (8)$$

where  $\Phi(z) = 1 + \sqrt{\pi} z e^{z^2} \operatorname{erfc}(-z)$ , and  $\operatorname{erfc}(\cdot)$  represents the complementary error function.

- **M-ary PAM**

For the  $M$ -ary PAM signal set, the conditional pdf of the received signal, assuming a normalized Rayleigh fading channel with no CSI is given by

$$p(y|x) = \frac{e^{-(y^2/\hat{\sigma}^2)}}{\sqrt{\pi}\hat{\sigma}^3} \left( \sqrt{\pi} x y \operatorname{erfc}\left(-\frac{\sqrt{2}xy}{2\sigma\hat{\sigma}}\right) + \sqrt{2}\sigma\hat{\sigma} e^{-\frac{x^2 y^2}{2\sigma^2 \hat{\sigma}^2}} \right), \quad (9)$$

where  $\hat{\sigma} = \sqrt{x^2 + 2\sigma^2}$ . Replacing (9) in (7), the true LLR values can be calculated.

- **M-ary QAM**

For the  $M$ -ary QAM constellation, assuming the normalized Rayleigh fading channel with no CSI at the receiver, we have the following conditional pdf of the received complex value  $y = y_r + jy_i$  given that the symbol  $x = x_r + jx_i$  is transmitted:

$$p(y|x) = \left( \frac{1}{\sqrt{2\pi}} (y_r x_r + y_i x_i) e^{\frac{x_r x_i y_r y_i}{\sigma^2 \hat{\sigma}^2}} \operatorname{erfc}\left(-\frac{\sqrt{2}(x_r y_r + x_i y_i)}{2\sigma \hat{\sigma}}\right) + \frac{\sigma \hat{\sigma}}{\pi} e^{-\frac{(x_r^2 y_r^2 + x_i^2 y_i^2)}{2\sigma^2 \hat{\sigma}^2}} \right) \frac{1}{\sigma \hat{\sigma}^3} e^{-\frac{\gamma^2}{2\sigma^2 \hat{\sigma}^2}}, \quad (10)$$

where  $\gamma^2 = x_r^2 y_i^2 + x_i^2 y_r^2 + 2\sigma^2(y_r^2 + y_i^2)$  and  $\hat{\sigma} = \sqrt{x_r^2 + x_i^2 + 2\sigma^2}$ .

True LLR values are then calculated using (7) and (10).

In this paper, our main focus is on Case 2, where the CSI is unknown at the receiver. In this case, the relationship between the channel LLR(s) and the channel output, given by (7), is rather complex. This means that the calculation of channel LLR values, required at the receiver, as a function of channel outputs is computationally expensive. Moreover, it would be very challenging to obtain the pdf of the channel LLR using (7). This pdf would be helpful to analyze the performance of different detection/decoding algorithms or to design one. In [6], the linear approximation of (1) was proposed to simplify the relationship between  $L$  and  $Y$  for the BPSK modulation. This approximation however has proved to be rather inaccurate resulting in performance degradations of a few tenths of a dB compared to true LLR calculations [14]. Recently, the more accurate linear approximation of (2) was proposed in [14]. This approximation was shown in [14] to perform well with binary LDPC codes and the BPSK modulation, and to result in rather large performance improvements compared to the approximation of (1). The downside however is the complicated relationship between  $\alpha_{\hat{C}}$  and  $\sigma^2$ , and the relatively high computational complexity of solving (3). The approach of [14] was then generalized in [15] to non-binary constellations, where piece-wise linear approximations of LLR were devised. These approximations performed very closely to true LLR calculations for an LDPC-coded BICM scheme based on the 8-PAM constellation, but relatively poorly for the 16-QAM constellation (a gap of about 0.2 dB to the true LLR calculations).

It is important to note that other approximations of the channel LLR for fading channels are given in the literature. For example, in [13], the *log-sum approximation*  $\log \sum_k \beta_k \approx \max_k \log \beta_k$  was used to approximate (5) as a piece-wise linear function. The approximation however is only good in the high SNR regime, where the sum is dominated by a single large term. Moreover, in the absence of CSI, which is of interest in this paper, the approximation does not lead to a piece-wise linear function and is much more complicated to implement.

Also noteworthy is that the LLR approximation (1) of [6] for BPSK with unknown CSI can be interpreted



as the minimum mean square error (MMSE) estimation of (6), the BPSK LLR for known CSI, given the received value  $y$ . The idea of using the MMSE estimate of LLR for known CSI as the approximate LLR in the absence of CSI can also be applied to non-binary modulations. Our study however shows that the performance degradation compared to true LLR calculations is rather large, e.g., about 0.5 dB for both 8-PAM and 16-QAM.

### B. LDPC Coding

We consider the application of binary LDPC codes for the transmission of information over the binary-input uncorrelated flat Rayleigh fading channel described in Subsection II-A. This channel is *memoryless*. It is also *output-symmetric* [9] when BPSK modulation is used for the transmission. It is however known that when  $M > 2$ , the  $m$  bit-channels associated with the BICM scheme are not necessarily output-symmetric [7]. To simplify the analysis, we thus use the technique of augmenting the bit-channels with i.i.d. channel adapters as in [7]. This makes the resulting channels output-symmetric. We also consider *symmetric message-passing decoders* [9] such that the conditional error probability is independent of the transmitted codeword [9]. For simplicity, therefore, we assume that the all-zero codeword is transmitted. This will be particularly helpful for density evolution, where the pdf of LLR values and their approximations is needed under this assumption.

The error rate performance of the transmission schemes is measured as a function of the channel signal-to-noise ratio (SNR), given by  $E_b/N_0$  for BPSK, and  $E_s/N_0$  for non-binary modulations, where  $E_b$  and  $E_s$  are the average energy per information bit and per transmitted symbol, respectively, and  $N_0$  is the one-sided power spectral density of the additive white Gaussian noise (AWGN). For BPSK, assuming  $X = \pm 1$  is transmitted, the SNR is related to the noise variance by  $E_b/N_0 = 1/(2R\sigma^2)$ , where  $R$  is the rate of the LDPC code. For 8-PAM and 16-QAM, given in Fig. 1, the SNR is given by  $21/(2\sigma^2)$ , and  $10/(2\sigma^2)$ , respectively.

To investigate the performance of transmission schemes, we use Monte-Carlo simulations at finite block lengths, and density evolution for asymptotic analysis. In the latter case, the threshold [9], [6], [7] of the transmission scheme for LDPC code ensembles is calculated as the measure of performance. For more information on LDPC codes and the calculation of the threshold, the reader is referred to [9], [6], [7].

### III. LLR APPROXIMATION BASED ON TAYLOR SERIES

#### A. Brief review of Taylor series

In this subsection, we provide the definition of one- and two-dimensional Taylor series (polynomials), which we subsequently use for the LLR approximation of one-dimensional and two-dimensional signal constellations, respectively.

**Definition 1.** Suppose that  $f(x)$  has  $n$  derivatives at a point  $x_0 \in [a, b]$ . The one-dimensional Taylor polynomial of order  $n$  for  $f(x)$  at point  $x_0$  is then defined by

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \end{aligned} \quad (11)$$

**Definition 2.** Suppose that  $f(x, y)$  has up to  $n$ th partial derivatives at a point  $(x_0, y_0)$  on a convex subset  $\Omega$  of  $\mathbb{R}^2$ . The two-dimensional Taylor polynomial of order  $n$  for  $f(x, y)$  at  $(x_0, y_0)$  is then defined by

$$P_n(x, y) = \sum_{\substack{\ell \geq 0 \\ \ell+m \leq n}} \sum_{m \geq 0} \frac{\partial^{m+\ell} f}{\partial x^\ell \partial y^m}(x_0, y_0) \frac{(x - x_0)^\ell}{\ell!} \frac{(y - y_0)^m}{m!} \quad \text{for any } (x, y) \in \Omega. \quad (12)$$

The difference between the true value of the function and its Taylor polynomial approximation is called the *remainder*. There are different forms to represent the remainder. The most commonly used is the *Lagrange form*, where the remainder in the case of one- and two-dimensional Taylor series, is given by [2]

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \quad (13)$$

and

$$\sum_{\substack{\ell \geq 0 \\ \ell+m=n+1}} \sum_{m \geq 0} \frac{\partial^{m+\ell} f}{\partial x^\ell \partial y^m}(x_1, y_1) \frac{(x - x_0)^\ell}{\ell!} \frac{(y - y_0)^m}{m!}, \quad (14)$$

respectively, where  $c$  is a number between  $x_0$  and  $x$ , and  $(x_1, y_1)$  is a point on the line connecting  $(x_0, y_0)$  and  $(x, y)$ . For the one-dimensional case, this requires  $f^{(1)}, f^{(2)}, \dots, f^{(n)}$  to be continuous on  $[a, b]$ , and that  $f^{(n+1)}$  to exist in  $(a, b)$  [2]. For the two-dimensional case, to have (14) as the remainder, the requirement is that  $f$  must have continuous partial derivatives of order  $n+1$  in a neighborhood of every point on a line segment joining two points  $(x_0, y_0)$  and  $(x, y)$  in  $\Omega$  [2]. More details on the convergence properties of Taylor series can be found in [2], [4], [5].

## B. LLR approximations

Our goal in this part of the paper is to approximate the LLR for one- and two-dimensional signal constellations as a function of the channel output, using Taylor polynomials of different orders. For the channel model considered here, the LLR is a differentiable function of the channel output and thus lends itself well to the Taylor series approximation. We are interested in using polynomials of smallest order as long as the approximation is accurate enough for the application under consideration. In particular, linear approximations are of most interest followed by second, third and larger order approximations. To obtain a sufficiently accurate approximation over a wide range of channel outputs and for a relatively low complexity, it is sometimes beneficial to use piece-wise approximations, where the domain of interest is partitioned into sub-domains. In each sub-domain then, a different approximation will be used. While it is possible to approximate the LLR within a given domain of interest with an arbitrarily high accuracy, if one has no constraints on the number of sub-domains and the order of approximations; in practice, due to the limited computational resources, the number of sub-domains and the order of approximations need to be small.

In addition to the selection of the number and the boundaries of the sub-domains, and the order of approximation in each sub-domain, one also needs to choose the point within each sub-domain at which the Taylor series is derived. This in general, could be a complicated optimization problem, defeating the whole purpose of finding low-complexity approximations for LLR. In this work, we limit ourselves to (piece-wise) linear, second and third order approximations. Since the accuracy of the LLR is particularly important for smaller LLR values, where a small error can change the sign of the LLR and thus the corresponding hard-decision, we select the roots of the LLR as the points around which the Taylor series is derived. The number and boundaries of the sub-domains will then be identified based on the number of roots, and the general shape of the LLR function. This will be explained in more details for BPSK, 8-PAM and 16-QAM in the following.<sup>2</sup> In particular, the simplicity of BPSK modulation makes it possible to derive closed-form analytical expressions for the Taylor approximations for any value of the channel SNR.

1) *BPSK Modulation:* The uncorrelated flat Rayleigh fading channel with BPSK modulation is output-symmetric. For an output-symmetric channel, LLR is an odd function of the channel output [14], i.e., if  $L = g(Y)$ , then  $g(-y) = -g(y)$ ,  $\forall y$ . It is thus natural to look for approximations which maintain the same odd symmetry. Both linear approximations (1) and (2) are odd symmetric. From (8), it is easy to

<sup>2</sup>BPSK, 8-PAM and 16-PSK are all used in [14], and [15]. We thus use the same constellations for comparison.

see that for the channel under consideration, the LLR is a continuous and differentiable function of the channel output with only a single root at  $y = 0$ . In fact, the Taylor series of (8) in the neighborhood of  $y = 0$  is given by

$$\sqrt{\frac{2\pi}{1+2\sigma^2}} \frac{y}{\sigma} + \frac{\sqrt{2\pi}(\pi-3)}{6(1+2\sigma^2)^{3/2}} \left(\frac{y}{\sigma}\right)^3 + O(y^5). \quad (15)$$

Corresponding to (15), we propose the following linear approximation of the channel LLR:

$$\hat{L}_{LT} = \sqrt{\frac{2\pi}{1+2\sigma^2}} \frac{Y}{\sigma} \triangleq \alpha_T Y, \quad (16)$$

and the non-linear approximation given by

$$\hat{L}_{NT} = \sqrt{\frac{2\pi}{1+2\sigma^2}} \frac{Y}{\sigma} + \frac{\sqrt{2\pi}(\pi-3)}{6(1+2\sigma^2)^{3/2}} \left(\frac{Y}{\sigma}\right)^3 \triangleq \alpha_T Y + \beta_T Y^3. \quad (17)$$

Note that for small channel SNR values where  $2\sigma^2 \gg 1$ , the approximation (16) reduces to (1).

In Fig. 2, true LLR values from (8) are compared with the proposed approximations (16) and (17) for  $\sigma = 0.6449$ . As expected, the approximations are very accurate for smaller values of  $y$ , with the non-linear approximation being more accurate and almost identical to the true LLR values for  $|y| < 4$ .

To apply density evolution to iterative algorithms with the approximated channel LLR values, we need to derive the pdf of  $\hat{L}_{LT}$  and  $\hat{L}_{NT}$  given in (16) and (17), respectively, assuming  $X = +1$  is transmitted. From (5), we have

$$p(y|+1, a) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right). \quad (18)$$

Averaging (18) over the distribution of  $A$ , we obtain

$$p(y|+1) = \frac{\sqrt{2/\pi}\sigma}{1+2\sigma^2} \exp\left(-\frac{y^2}{1+2\sigma^2}\right) \Theta\left(\frac{y}{\sqrt{2\sigma^2(1+2\sigma^2)}}\right), \quad (19)$$

where  $\Theta(z) = e^{-z^2} + \sqrt{\pi}z \operatorname{erfc}(-z)$ . Based on (16), the pdf of  $\hat{L}_{LT}$ , given that  $X = +1$  is transmitted, is derived as:

$$p_{\hat{L}_{LT}\{\sigma\}}(\hat{l}) = \frac{\sigma^2}{\pi \sqrt{1+2\sigma^2}} \left( \exp\left(-\frac{(1+2\sigma^2)\hat{l}^2}{4\pi}\right) + \frac{1}{2}\hat{l} \exp\left(-\frac{\sigma^2 \hat{l}^2}{2\pi}\right) \operatorname{erfc}\left(-\frac{\hat{l}}{2\sqrt{\pi}}\right) \right). \quad (20)$$

The pdf of  $\hat{L}_{NT}$  is calculated using (19), and based on the relationship (17) between  $\hat{L}_{NT}$  and  $Y$  as follows:

$$p_{\hat{L}_{NT}\{\sigma\}}(\hat{l}) = \frac{p(y|+1)\big|_{y=g^{-1}(\hat{l})}}{|g'(y)|\big|_{y=g^{-1}(\hat{l})}}, \quad (21)$$

where  $g(y) = \alpha_T y + \beta_T y^3$  with the derivative  $g'(y) = \alpha_T + 3\beta_T y^2$ , and the inverse

$$g^{-1}(\hat{l}) = \frac{\phi(\hat{l})}{6\beta_T} - \frac{2\alpha_T}{\phi(\hat{l})}, \quad (22)$$

in which  $\phi(\hat{l}) = \sqrt[3]{12\beta_T^2(9\hat{l} + \sqrt{\frac{12\alpha_T^3}{\beta_T} + 81\hat{l}^2})}$ .

2) *8-PAM*: Consider the 8-PAM signal set with Gray labeling as shown in Fig. 1. True LLR values for the three bits can be calculated using (7) and (9). These values clearly depend on the value of the channel SNR. We now explain the Taylor approximation of the three LLRs based on a given value of SNR= 7.91 dB. In Fig. 3, for SNR= 7.91 dB, the true LLR values are plotted with full lines as a function of the channel output  $y$ . As expected, all three functions are symmetric with respect to the vertical axis at  $y = 0$ . The LLR of the first bit,  $l^{(1)}$ , has a single root at  $y = 0$ . The roots of  $l^{(2)}$  and  $l^{(3)}$  are respectively  $\{-3.3449, 3.3449\}$  and  $\{-6.9832, -1.8848, 1.8848, 6.9832\}$ . We thus partition the domain of  $y$  values to one, two and four sub-domains, for the three LLRs, respectively.

Suppose that the  $k$ th derivative of the LLR function of the  $i$ th bit is denoted by  $f_i^{(k)}$ . For the first bit, a linear approximation of the LLR can be obtained by the first order Taylor polynomial at  $y = 0$  as follows:

$$\hat{L}^{(1)} = f_1^{(1)}(0)Y. \quad (23)$$

Due to the symmetry of the LLR functions, we have

$$\begin{aligned} f_i^{(2k-1)}(y) &= -f_i^{(2k-1)}(-y), \\ f_i^{(2k)}(y) &= f_i^{(2k)}(-y), \end{aligned} \quad (24)$$

for integers  $k \geq 1$ . We thus have the following piece-wise linear approximations based on Taylor series for the second and the third bits, respectively:

$$\hat{L}^{(2)} = f_2^{(1)}(y_1)(Y - y_1)I(Y \geq 0) - f_2^{(1)}(y_1)(Y + y_1)I(Y \leq 0) = f_2^{(1)}(y_1)|Y| - f_2^{(1)}(y_1)y_1, \quad (25)$$

$$\begin{aligned} \hat{L}^{(3)} &= f_3^{(1)}(y_2)(Y - y_2)I(c \leq Y) + f_3^{(1)}(y_3)(Y - y_3)I(0 \leq Y < c) \\ &\quad - f_3^{(1)}(y_3)(Y + y_3)I(-c < Y \leq 0) - f_3^{(1)}(y_2)(Y + y_2)I(Y \leq -c) \\ &= \left(f_3^{(1)}(y_2)|Y| - f_3^{(1)}(y_2)y_2\right)I(c \leq |Y|) + \left(f_3^{(1)}(y_3)|Y| - f_3^{(1)}(y_3)y_3\right)I(|Y| < c). \end{aligned} \quad (26)$$

where  $I(\cdot)$  is the indicator function,  $y_1 = 3.3449$ ,  $y_2 = 6.9832$ ,  $y_3 = 1.8848$ , and  $c = 3.7266$  is the  $y$  value of the point where the two linear approximations for the LLR function of the third bit intersect in the  $y > 0$  region (see Fig. 3c).

In Table I, the coefficients of the linear approximation (23) and the piece-wise linear approximations (25) and (26) for SNR= 7.91 dB are given in the first row. It is worth mentioning that for all three functions, second order derivatives at the roots of LLR functions are zero. The coefficients of the terms with degree 3 in the Taylor polynomial are also given in the second row of the table.

In Fig. 3, the first, and the third order Taylor polynomials for the approximation of the three LLRs are also plotted. Comparison with true LLRs shows that the accuracy of the approximations improve consistently by increasing the order of the polynomials.

3) 16-QAM: Consider the 16-QAM constellation with Gray labeling as shown in Fig. 1. True LLR values for the four bits can be calculated using (7) and (10). These values clearly depend on the value of the channel SNR. We now explain the Taylor approximation of the four LLRs based on the value of SNR= 4.89 dB.

Since 16-QAM is two-dimensional, we need to use two-dimensional Taylor polynomials for the approximation of LLRs, where each LLR is a function of the channel output  $y = y_r + jy_i$ . One difference compared to the one-dimensional cases is that, the roots of the LLR functions in the two-dimensional case are located on one or more two-dimensional curves rather than belonging to a discrete set of values. To derive the Taylor polynomial in a sub-domain, a single point from such a curve within the sub-domain should then be selected so that the Taylor coefficients can be computed at that point. The selection of such a point depends on the general shape and symmetries of the LLR function, as explained in the following.

In Figures 5(a) and (b), the contours of fixed (true) LLR values for the first and the second bit of the constellation are shown in the  $(y_r, y_i)$  plane, respectively. As can be seen, for the first bit, the curve corresponding to  $l^{(1)} = 0$  is  $y_r = 0$ . For the second bit, there are two curves, symmetric with respect to  $y_r = 0$  corresponding to  $l^{(2)} = 0$ . The existing symmetries in the LLR functions with respect to both  $y_r = 0$  and  $y_i = 0$  suggest the selection of  $(0, 0)$ , and  $(\xi, 0)$  and  $(-\xi, 0)$ , as the points around which the Taylor approximations of  $l^{(1)}$  and  $l^{(2)}$ , should be derived, respectively, where  $\xi = 1.8908$  is the intersection of the curve corresponding to  $l^{(2)} = 0$  with the line  $y_i = 0$  in the region  $y_r > 0$  of the  $(y_r, y_i)$  plane.

The corresponding Taylor polynomials of the third and the second order for  $L^{(1)}$  and  $L^{(2)}$ , are respectively derived as:

$$\hat{L}^{(1)} = -0.9878 Y_r - 0.04285 Y_r Y_i^2 - 0.01654 Y_r^3, \quad (27)$$

$$\hat{L}^{(2)} = -0.9285 + 0.2690 |Y_r| + 0.1174 Y_r^2 - 0.0364 Y_i^2. \quad (28)$$

Based on the symmetry in the Gray labeling of 16-QAM, the LLR values for the third and the fourth bits are similar to those of the first and the second bits, respectively, except that the real and the imaginary

parts of  $y$  need to be switched. We thus have the following Taylor approximations for  $L^{(3)}$  and  $L^{(4)}$ , respectively:

$$\hat{L}^{(3)} = 0.9878 Y_i + 0.04285 Y_i Y_r^2 + 0.01654 Y_i^3, \quad (29)$$

$$\hat{L}^{(4)} = -0.9285 + 0.2690 |Y_i| + 0.1174 Y_i^2 - 0.0364 Y_r^2. \quad (30)$$

#### IV. NUMERICAL RESULTS AND DISCUSSION

In this paper, we mainly compare our results with those of [14] and [15], which are the best known approximations of the channel LLR for uncorrelated flat Rayleigh fading channels in terms of performance. Similar to [14] and [15], we consider the three modulations BPSK, 8-PAM and 16-QAM. Our results however, can be easily extended to other linear modulations.

To obtain the coefficients of Taylor approximations, we resort to the asymptotic analysis of density evolution with the SNR threshold as the performance criterion. The goal is then to find sufficiently accurate Taylor approximations for the LLR functions so that the resulting threshold of the BICM scheme is close to the threshold obtained using the true LLR values. As the Taylor coefficients are functions of the channel SNR, one approach would be to start from an SNR value above the threshold, and find the corresponding Taylor coefficients. Then find the threshold corresponding to the resulting Taylor approximation. (This threshold will be smaller than the starting SNR.) Use the new SNR threshold and find the corresponding Taylor approximation. Use this new approximation to find the next threshold. Continue this process until it converges to a fixed point, i.e., the Taylor coefficients and the SNR threshold remain unchanged in two successive iterations. A simpler approach, with slightly inferior results, is to find the SNR threshold using the true LLR values and then use that value of SNR to find the Taylor coefficients.

##### A. BPSK

In this part, we present analysis and design results based on the proposed linear and non-linear LLR approximations for the BPSK modulation through a number of examples.

**Example 1.** *In this example, decoding thresholds<sup>3</sup> of two regular ensembles of LDPC codes based on different LLR approximations over the uncorrelated flat Rayleigh fading channel are calculated. The ensembles have the following degree distributions:  $\lambda_1(x) = x^2$ ,  $\rho_1(x) = x^5$ ;  $\lambda_2(x) = x^3$ ,  $\rho_2(x) =$*

<sup>3</sup>The thresholds are given in terms of  $E_b/N_0$ .

$x^{15}$ .<sup>4</sup> The results corresponding to the two ensembles are given in Tables II and III, respectively. These results indicate that while there is a large performance gap between the linear approximations (1) and (2), the performance of the proposed linear approximation (16) is practically identical to that of (2). Unlike (16), however, the computational complexity of (2) is relatively high. Also noteworthy is the fact that the proposed approximation is analytical while the approximation (2) of [14] must be obtained numerically. In both tables, the results of optimal linear approximation ( $\hat{L}_{opt} = \alpha_{opt}Y$ ),<sup>5</sup> and the non-linear approximation of (17) are also given. The results show that both linear approximations perform practically optimally for these codes. Our proposed non-linear approximation performs the same as the optimal linear approximation and does not provide any further improvement.

**Example 2.** In this example, decoding thresholds for two irregular LDPC code ensembles are calculated. The first ensemble has rate  $1/2$  and is optimized for a normalized Rayleigh fading channel with known CSI (first code in Table I of [6]). Using approximations (1) and (2), the  $E_b/N_0$  threshold for unknown CSI is 3.74 dB and 2.98 dB, respectively. For the proposed linear and non-linear approximations, the thresholds are 2.98 dB and 2.97 dB, respectively.

The second ensemble is a rate- $1/2$  threshold optimized ensemble for normalized Rayleigh fading channel with unknown CSI with approximation (2) (Code 2 of [14]). Again, for this ensemble, both (2) and the proposed linear approximation have the same threshold of 2.76 dB. The threshold for the proposed non-linear approximation however is improved to 2.73 dB.

To compare the error rate performance of the proposed linear approximation and that of [14] at finite block lengths, we have tested a number of regular and irregular LDPC codes. For all cases, the two approximations performed practically the same, and very close to the performance with true LLR calculations. One such example can be found in [1].

**Example 3.** In this example, we construct irregular LDPC code ensembles optimized for uncorrelated flat Rayleigh fading channels with unknown CSI based on the proposed non-linear approximation (17). To fairly compare our results with those of [14], we choose the exact same constituent degrees as those in similar examples of [14], given in Table III of [14]. We design two ensembles, one over a channel with  $\sigma = 0.7436$  where we maximize the rate, and the other with the fixed rate of  $1/2$  where we optimize the

<sup>4</sup>The threshold results presented in this paper are obtained for an 11-bit iterative belief propagation decoder [9] with the maximum number of iterations 1000. The target probability of error for variable node to check node message error rate is set at  $10^{-7}$ . The maximum LLR is chosen to be 25 except for simulations related to non-linear Taylor approximation, for which the maximum LLR of 35 is used.

<sup>5</sup> $\alpha_{opt}$  is obtained by exhaustive search using density evolution.



*threshold. Similar to the nomenclature used in [14], these ensembles are labeled as “Code 1” and “Code 2,” respectively. The degree distributions for the two ensembles are given in Table IV. Code 1 has a rate of 0.4941 compared to 0.4937 of the similar code in [14]. The threshold of Code 2 is 2.68 dB which is 0.08 dB better than the similar result of [14].*

### B. 8-PAM

Similar to [15], in this part, we consider a BICM scheme with Gray labeled 8-PAM signal set, as shown in Fig. 1, along with a  $(3, 4)$ -regular LDPC code  $((\lambda(x) = x^2, \rho(x) = x^3))$ . The decoding threshold for this scheme using true LLR values is 7.85 dB. Using this value of SNR and by following the general piece-wise linear approximation described in Subsection III-B2, we then obtain the required Taylor coefficients. Using this LLR approximation, the decoding threshold is degraded to 7.92 dB. If we update the coefficients of the approximation based on SNR= 7.92 dB, the new threshold will be 7.91 dB. The next set of Taylor coefficients, obtained based on SNR= 7.91 dB, however, do not change the threshold. This SNR value (7.91 dB) is what we used in Subsection III-B2 to derive the Taylor approximations for the LLR functions of 8-PAM. We refer the reader to Subsection III-B2 for the details of the Taylor approximations.

The decoding threshold of the BICM scheme using the first and the third order Taylor polynomials are 7.91 dB and 7.86 dB, respectively. This can be compared to the threshold obtained using the true LLRs, 7.85 dB, and the one obtained in [15], 7.88 dB.

To evaluate the finite length performance of the BICM scheme, we randomly construct a  $(3, 4)$ -regular LDPC code of length 12,000 and girth 6. The bit error rate (BER) performance of this code with the 8-PAM constellation over the uncorrelated flat Rayleigh fading channel is shown in Fig. 4 for both the first and the third order Taylor approximations. Belief propagation is used for the decoding of the LDPC code with the maximum number of iterations 100. In Fig. 4, we have also included the BER performance of the same scheme with true LLR values, and the piece-wise linear approximation of [15] for LLRs. It is seen that the BER performance of our piece-wise linear approximation is similar to the piece-wise linear approximation of [15], and almost identical to the performance of the more complex true LLRs. This is while the derivation of our approximation is much simpler than that of [15]. It can be seen in Fig. 4 that no practically significant gain in performance is obtained by going from first order to the third order Taylor approximation. This however, is not the case for 16-QAM, as demonstrated in the next subsection.

### C. 16-QAM

In this part, similar to [15], we consider the 16-QAM signal set with Gray labeling shown in Fig. 1, and a  $(3, 4)$ -regular LDPC code in the BICM scheme. To obtain the Taylor approximations, we use the general approach described in Subsection III-B3. A Piece-wise linear approximation results in about 0.2 dB degradation compared to true LLR values, as also observed in [15]. We thus consider higher order Taylor polynomials. The BICM scheme has an SNR threshold of 4.83 dB with true LLR values. Based on this SNR value, we then obtain the Taylor coefficients of the third order approximations for bits one and three and the second order approximations for bits two and four. These approximations are then used to find the new threshold of the scheme. Repeating this process, we finally converge to the approximations given in (27)-(30) for the four LLRs and the SNR threshold of 4.89 dB which is very close to that of true LLRs, 4.83 dB. If we consider the third order Taylor polynomial (instead of the second order) for the second and forth bits, the threshold improves slightly to 4.87 dB. Note that this provides 0.15 dB improvement over the 5.02 dB threshold obtained by the piece-wise linear approximation of [15] for this scheme. This is while the derivation of the proposed approximation is also simpler than that of [15].

Finite block length BER performance of the BICM scheme using the Taylor approximations for the LLRs is shown in Fig. 4. The results are for a randomly constructed regular  $(3, 4)$ -LDPC code with length 12000 and girth 6. The decoding algorithm is belief propagation with maximum number of iterations 100. For comparison, BER curves for the true LLR values, and the piece-wise linear approximation of [15] are also given in the figure. As can be seen, the proposed Taylor approximation performs almost the same as true LLRs, and outperforms the approximation of [15] by about 0.2 dB. This is while the complexity of computing the Taylor approximations is much less compared to both the true LLR calculation and the piece-wise linear approximation of [15].

Although the results reported in this paper are obtained for a fading channel with known channel SNR at the receiver, they can also be applied to the case where such information is unavailable at the receiver. In such a case, if the coding scheme is given and has a threshold  $\sigma^*$  with the proposed Taylor approximation, one would use the proposed approximate LLR values by substituting  $\sigma^*$  in the corresponding Taylor approximation. On the other hand, if one is interested in the design of a coding scheme, such as an irregular ensemble of LDPC codes of a given rate, over a channel with unknown  $\sigma$  based on the proposed approximations, one can perform the design, assuming that  $\sigma$  is known, to optimize the threshold. If the threshold value is  $\sigma^*$ , one should then use the LLR approximation by substituting  $\sigma$  with  $\sigma^*$ .

## V. CONCLUSION

In communication systems, the receiver often requires to calculate the channel LLR for the processing of the received signal. Over wireless channels, this will have to be performed almost always at the absence of the full knowledge of CSI. Under such conditions, the calculation of true LLR values is computationally expensive. Approximations of LLR, are thus important to find. In this paper, we proposed simple (piece-wise) linear and non-linear approximations of channel LLR based on Taylor Series. For the uncorrelated flat Rayleigh fading channel using one-dimensional linear modulations in the context of LDPC-coded BICM schemes, the proposed (piece-wise) linear approximations perform practically the same as the best known (piece-wise) linear approximations of [14] and [15] and very close to the performance of a scheme using true LLR values. This is while the derivation of the proposed Taylor approximations for the LLR is simpler than the computation of the approximations in [14] and [15], and significantly simpler than the calculation of true LLR values. For two-dimensional constellations, where the piece-wise linear approximation of [15] causes non-negligible performance loss compared to the true LLR calculations, our proposed Taylor approximations still perform very close to true LLR calculations with significantly lower complexity.

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TABLE I: COEFFICIENTS OF TAYLOR POLYNOMIALS FOR THE LLR APPROXIMATIONS OF 8-PAM

	Bit 1	Bit 2	Bit 3
Degree-1 coef.	$f_1^{(1)}(0) = 1.2135$	$f_2^{(1)}(3.3449) = -0.6147$	$f_3^{(1)}(6.9832) = -0.3419, f_3^{(1)}(1.8848) = 0.6046$
Degree-3 coef.	$f_1^{(3)}(0) = 0.1420$	$f_2^{(3)}(3.3449) = 0.0039$	$f_3^{(3)}(6.9832) = -0.0070, f_3^{(3)}(1.8848) = -0.2920$

TABLE II: THRESHOLDS FOR THE LDPC ENSEMBLE WITH  $\lambda_1(x) = x^2$ ,  $\rho_1(x) = x^5$  UNDER BELIEF PROPAGATION WITH DIFFERENT LLR APPROXIMATIONS

	$\alpha_A = 4.514$	$\alpha_{\hat{C}} = 2.957$	$\alpha_T = 2.874$	$\alpha_{opt} = 2.957$	Nonlinear Taylor
$\sigma_z^*$	0.6266	0.6449	0.6445	0.6449	0.6449
$\frac{E_b}{N_0}^*$ (dB)	4.06	3.81	3.82	3.81	3.81

TABLE III: THRESHOLDS FOR THE LDPC ENSEMBLE WITH  $\lambda_2(x) = x^3$ ,  $\rho_2(x) = x^{15}$  UNDER BELIEF PROPAGATION WITH DIFFERENT LLR APPROXIMATIONS

	$\alpha_A = 15.616$	$\alpha_{\hat{C}} = 6.302$	$\alpha_T = 6.054$	$\alpha_{opt} = 6.287$	Nonlinear Taylor
$\sigma_z^*$	0.3369	0.3677	0.3674	0.3677	0.3677
$\frac{E_b}{N_0}^*$ (dB)	7.69	6.93	6.94	6.93	6.93

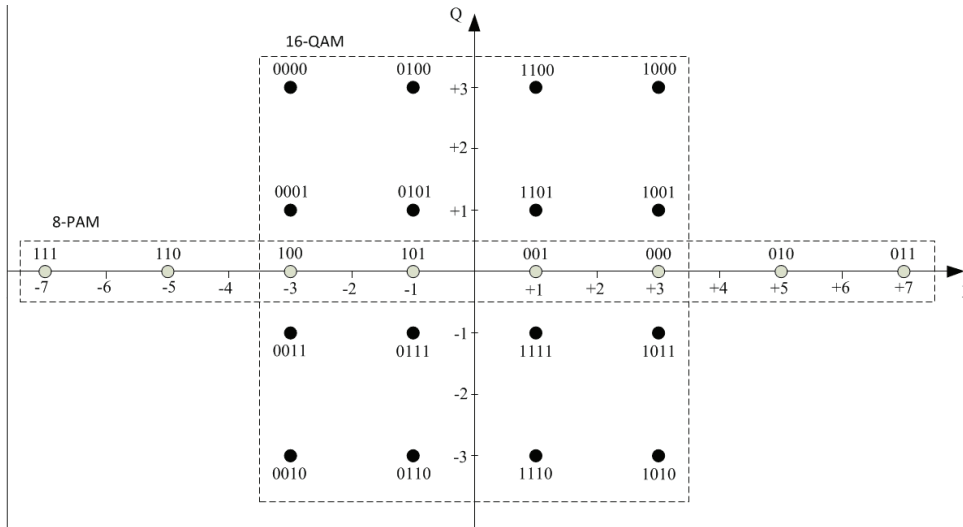


Fig. 1: 8-PAM and 16-QAM constellations with Gray labeling

TABLE IV: LDPC CODES DESIGNED FOR THE UNCORRELATED FLAT RAYLEIGH FADING CHANNEL WITH UNKNOWN CSI BASED ON APPROXIMATION (17). CODE 1 AND 2 ARE RATE- AND THRESHOLD-OPTIMIZED, RESPECTIVELY.

	Code 1	Code 2
$\lambda_2$	0.20525	0.20683
$\lambda_3$	0.21067	0.21646
$\lambda_4$	0.00037	0.00046
$\lambda_5$		0.00075
$\lambda_6$	0.00180	0.00230
$\lambda_7$	0.18140	0.12574
$\lambda_8$	0.07439	0.13858
$\lambda_9$	0.00248	0.00343
$\lambda_{10}$	0.00099	0.00137
$\lambda_{11}$	0.00059	0.00081
$\lambda_{15}$	0.00026	0.00035
$\lambda_{20}$	0.00024	0.00032
$\lambda_{29}$	0.00181	0.00247
$\lambda_{30}$	0.31975	0.30013
$\rho_9$	1.0000	1.0000
<b>Rate</b>	0.4941	0.5000
$\sigma^*$	0.7436	0.7345
$E_b/N_0^* (dB)$	2.63	2.68

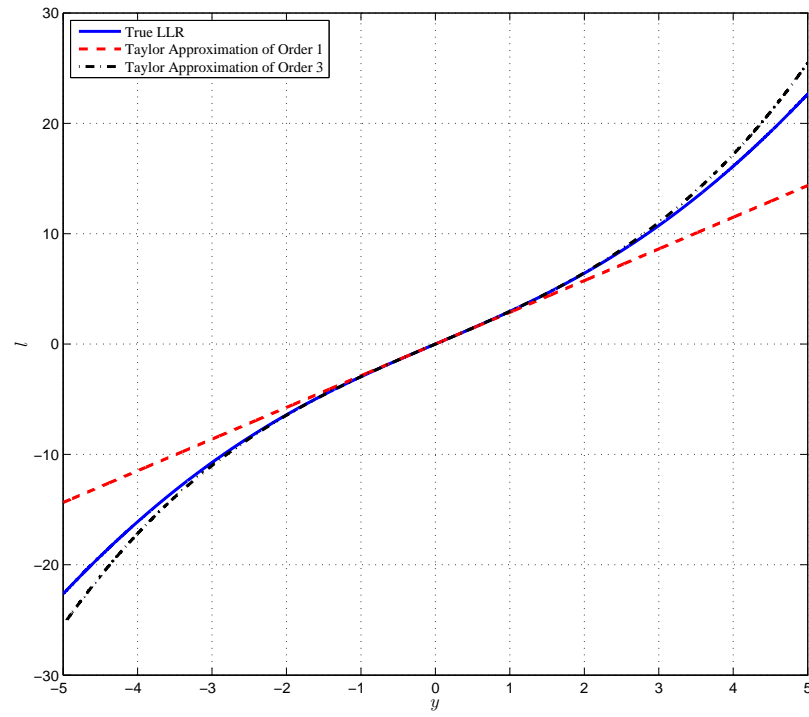


Fig. 2: Comparison of true LLR values and the approximations obtained by Taylor series for the uncorrelated flat Rayleigh fading channel with unknown CSI and  $\sigma = 0.6449$ .

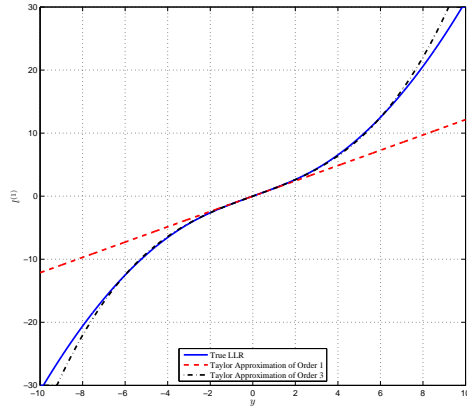
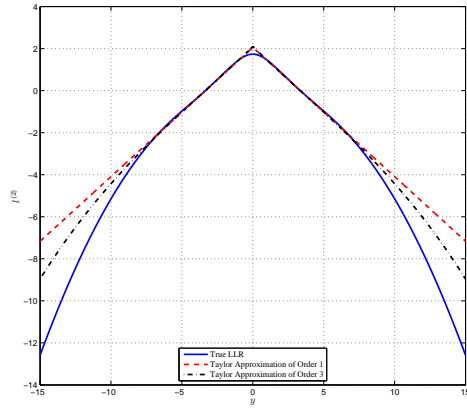
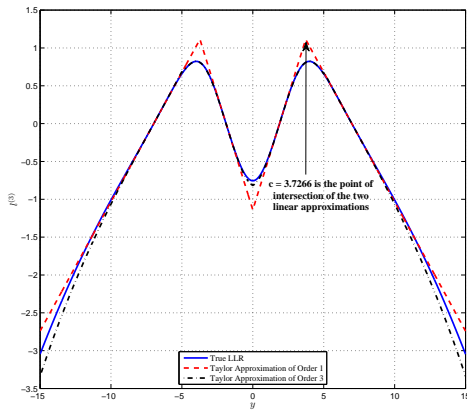
(a)  $i = 1$ (b)  $i = 2$ (c)  $i = 3$ 

Fig. 3: True bit LLR values  $l^{(1)}$ ,  $l^{(2)}$ , and  $l^{(3)}$  as functions of the channel output  $y$  for 8-PAM at SNR=7.91 dB, along with the corresponding (piece-wise) Taylor approximations.



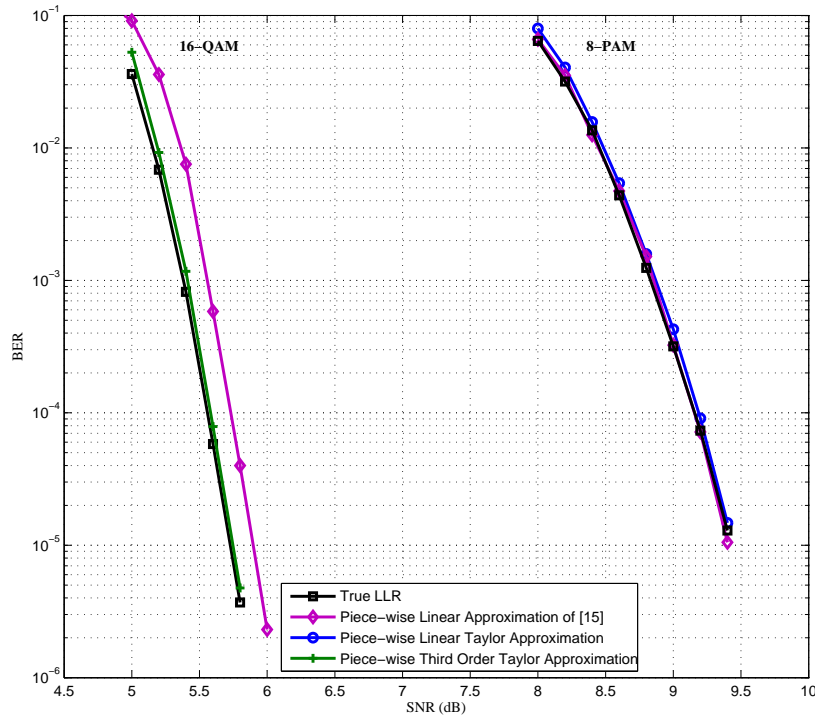


Fig. 4: BER performances of BICM schemes with 8-PAM and 16-QAM in combination with a  $(3, 4)$ -regular LDPC code of length 12000 based on various Taylor approximations of LLR, the piece-wise linear approximation of [15], and true LLR values, over the uncorrelated flat Rayleigh fading channel.

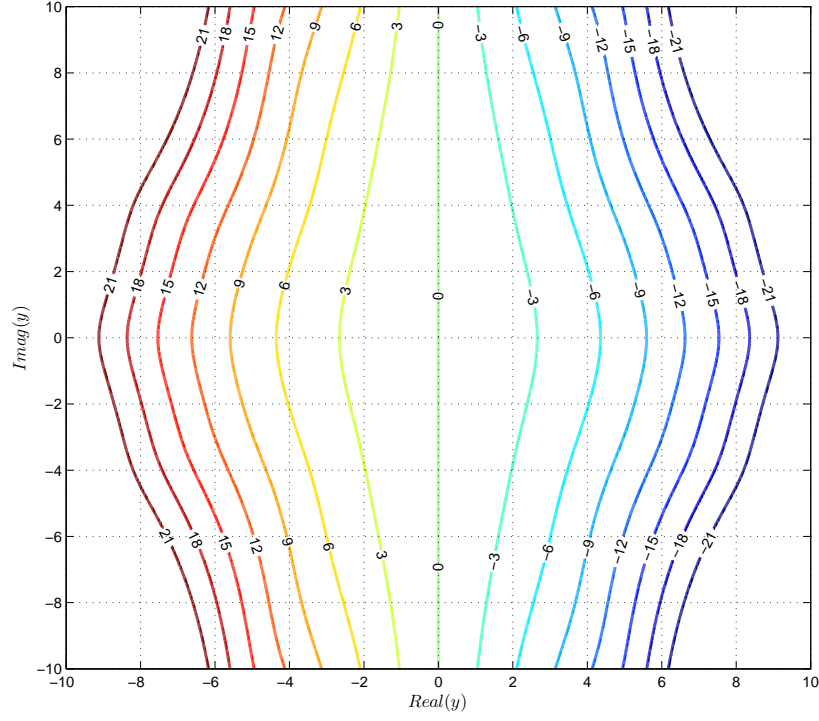
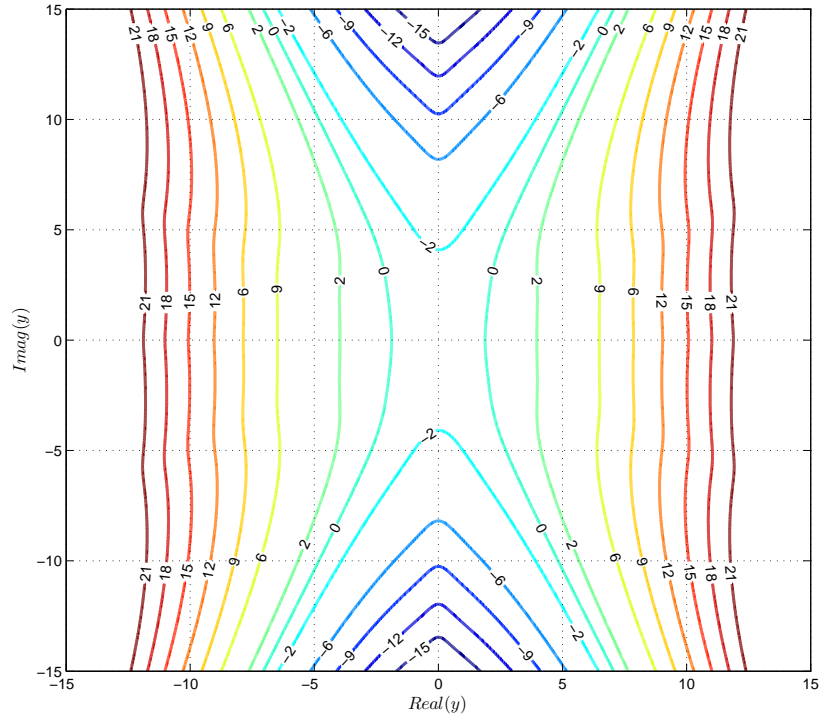
(a) Contours of fixed  $l^{(1)}$ (b) Contours of fixed  $l^{(2)}$ 

Fig. 5: Contours of fixed true bit LLR values  $l^{(1)}$  and  $l^{(2)}$  in the  $(y_r, y_i)$  plane for the 16-QAM at SNR=4.89 dB.